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# Some aspects of the initial-value problem for the inviscid motion of a contained, rotating, weakly-stratified fluid

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T he initial-value problem for the linear, inviscid motion of a contained, rotating stratified fluid is considered in the limit of weak stratification, that is, for small values of the stratification parameter $S = N^2/\Omega^2$, where $N$ is the Brunt–Väisälä frequency and $\Omega$ is the rotational frequency. The limiting flow is of interest because, although the initial-value problem has been studied, both for the case of a homogeneous, rotating fluid and for the case of a stratified, rotating fluid, the exact relationship of the two flows, in the limit of vanishing stratification, is not straightforward. For example, the method of determining, from the initial conditions, the steady geostrophic component of the flow of a rotating, stratified fluid does not in general give a motion that reduces, in the limit $S \to 0$, to the steady component of the flow of a homogeneous fluid. By including a consideration of slow unsteady motions that vary on a time scale dependent on the stratification parameter, the relationship of the limiting flow to the flow of a homogeneous fluid is established.

## 1. Introduction

A general linear theory of the initial-value problem for the motion of a contained, homogeneous, rotating, viscous, incompressible fluid has been given by Greenspan (1965). The theory describes the establishment, in a container of fairly general shape, of a state of rigid rotation from an initial state which is a small perturbation from rigid rotation. The Rossby number $\epsilon = U/\Omega L$ is therefore considered negligibly small and the approximate governing equations are linear. It is also assumed that the direct action of viscous forces is confined to thin boundary layers on the container surface. The inviscid interior flow is first determined from the initial data and then corrected for the effects of the action of the viscous forces in the boundary layers. A linear system of partial differential equations, with time-independent coefficients, governs the interior inviscid flow and the solution is described as a superposition of normal oscillatory modes and a steady geostrophic mode. The mean circulation theorem is proved and is used to determine the geostrophic steady component from the initial data.

The corresponding initial-value problem, for the *inviscid* motion of a contained, rotating fluid, which is also stratified in a gravity force field, has been studied by Howard & Siegmann (1969). A linear theory is developed to describe motions that are assumed to be small departures from a state of rigid rotation and hydrostatic equilibrium. The fluid is considered, in general, to be compressible, and no restriction is made on the spatial dependence of the gravitational field or on the container shape. The solution is sought as a superposition of a time-dependent component and a time-independent geostrophic flow, and the main results concern the method of determination of the steady geostrophic mode from the initial data.

Included as a special case in the study of Howard & Siegmann is, of course, the treatment of the initial-value problem for the inviscid motion of a contained, rotating, density-stratified liquid which obeys the Boussinesq approximation, and which is stratified in a gravity force field, where the gravity vector is anti-parallel to the rotation vector. We will restrict our attention, in this paper, to that situation. In particular, it has been shown by Howard & Siegmann that the steady geostrophic mode can be uniquely determined from the initial conditions by the solution to an equation resulting from the time-independence of the potential vorticity. Other temporally conserved quantities result in the proper boundary conditions. A point of interest, with which we will be concerned here, is that, in the limit of weak statification, i.e. for vanishingly small values of the stratification parameter $S = N^2/\Omega^2$, where $N$ is the Brunt–Väisälä frequency and $\Omega$ is the rotational frequency, the method of determining the steady mode for the stratified fluid does not, in general, reduce to that used in the problem with a homogeneous fluid. As a consequence, the exact relation of the limiting flow to the flow of a homogeneous fluid is not completely obvious. As mentioned in Howard & Siegmann, to relate the two flows, it is necessary to include a consideration of slow unsteady motions, with time scales dependent on the stratification parameter.

In this paper, we consider the initial-value problem for the inviscid motion of a contained, rotating, stratified fluid in the limit of vanishing stratification, and we primarily investigate the relation of the limiting flow to the steady geostrophic component of the flow of a homogeneous fluid. It turns out that the characteristics of the limiting flow are different for containers with closed contours of constant height (Greenspan 1965; also defined in § 2) that lie in planes perpendicular to the rotation and gravity vectors (we will refer to these as 'flat' contours), and for containers with a more general shape which, for the geostrophic flow of a homogeneous fluid, require a component of velocity parallel to the rotation and gravity vectors (we will refer to this case as 'sloping' contours). Containers with 'flat' contours are considered in § 3. For containers with 'sloping' contours, there is a further difference in the limiting flow when the contours are uniquely defined and when they are not. These two cases are treated in order in § 4.

An example of a container with a non-unique set of 'sloping' geostrophic contours is the 'doubly sliced' cylinder shown in figure 1. This configuration is formed from a circular cylinder, with axis aligned with the rotation vector and with plane parallel top and bottom surfaces that slope with respect to a plane perpendicular to the rotation vector. In this case, it is found that for $S \ll 1$ a set of low-frequency modes, which might be called contained weak stratification waves, arise to take the place of the geostrophic flow of a homogeneous fluid. The frequencies of these modes depend on $S$ and approach zero as $S \to 0$. It is shown in § 5 that these modes exist in other situations where a weak stratification interacts with the lowest-order geostrophic flow.

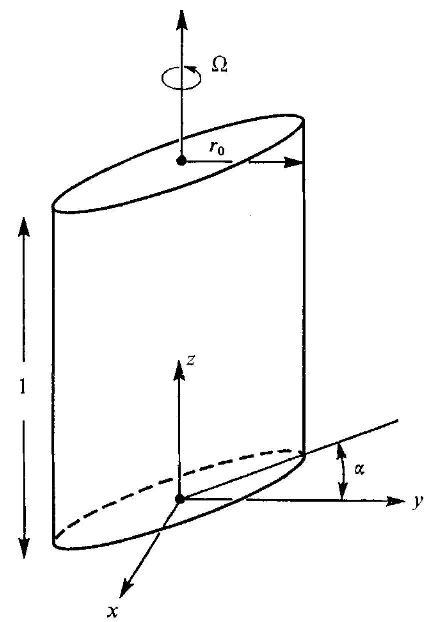


FIGURE 1. The 'doubly sliced' cylinder configuration where the angle $\alpha = \tan^{-1}(b/a)$.

## 2. Formulation

We initially consider a viscous, heat conducting, incompressible fluid, which satisfies the Boussinesq approximation, in a frame of reference rotating with a uniform angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$ and acted on by a gravitational acceleration $\mathbf{g} = -g\hat{\mathbf{k}}$ which is antiparallel to the rotation vector. The governing equations are

$$\nabla . \mathbf{q} = 0,$$

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}.\nabla\mathbf{q} + 2\Omega\hat{\mathbf{k}}\times\mathbf{q} = -\frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0}g\hat{\mathbf{k}} + \frac{1}{2}\frac{\rho}{\rho_0}\Omega^2\nabla|\hat{\mathbf{k}}\times\mathbf{r}|^2 + \nu\nabla^2\mathbf{q},$$

$$\frac{\partial T}{\partial t} + \mathbf{q}.\nabla T = \kappa\nabla^2 T,$$

$$\rho = \rho_0[1-\alpha(T-T_0)],$$

where $\mathbf{q}$, $p$, $\rho$ and $T$ are respectively the velocity, pressure, density, and temperature of the fluid at a point $\mathbf{r}$; $\nu$, $\kappa$ and $\alpha$ are respectively the constant kinematic viscosity, thermometric conductivity, and coefficient of thermal expansion; $\rho_0$ and $T_0$ are constant reference values of the density and temperature; $\hat{\mathbf{k}}$ is a constant unit vector in the $z$ direction in Cartesian $(x, y, z)$ or cylindrical polar $(r, \theta, z)$ co-ordinates.

We assume that the Froude number $F = \Omega^2 L/g$ is small and consider a linear equilibrium temperature and density distribution (see Greenspan 1968, § 1.4) given by

$$T_s = T_0 + \Delta T_0 z/L, \quad \rho_s = \rho_0(1 - \alpha \Delta T_0 z/L),$$

where $\Delta T_0(>0)$ is the basic temperature difference imposed over the height $L$.

The variables are non-dimensionalized in the following manner:

$$\mathbf{q} = U\mathbf{q}^*, \quad \mathbf{r} = L\mathbf{r}^*, \quad t = \Omega^{-1}t^*,$$

$$p = p_0 - \rho_0 gLz^* + \tfrac{1}{2}\rho_0 gL\alpha\Delta T_0 z^{*2} + \rho_0 U\Omega Lp^*,$$

$$T = T_s + \Delta TT^*, \quad \rho = \rho_s + \rho_0 \alpha \Delta T \rho^*,$$

where $p_0$ is a constant reference pressure, $U$ is a reference velocity, and $\Delta T$ is a reference perturbation temperature.

The resulting dimensionless equations are (dropping the asterisks)

$$\nabla . \mathbf{q} = 0, \tag{2.1a}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \epsilon \mathbf{q} . \nabla \mathbf{q} + 2\hat{\mathbf{k}} \times \mathbf{q} = -\nabla p - \delta\rho\hat{\mathbf{k}} + E\nabla^2 \mathbf{q}, \tag{2.1b}$$

$$\frac{\partial T}{\partial t} + \epsilon \mathbf{q} . \nabla T + \frac{S}{\delta}\mathbf{q} . \hat{\mathbf{k}} = \frac{E}{\sigma}\nabla^2 T, \tag{2.1c}$$

$$\rho = -T, \tag{2.1d}$$

where $E = \nu/\Omega L^2$ is the Ekman number, $\sigma = \nu/\kappa$ is the Prandtl number, $S = \alpha g \Delta T_0/\Omega^2 L$ is the stratification parameter (the inverse of the internal Froude number) and can be written as $S = N^2/\Omega^2$, where $N^2 = \alpha g \Delta T_0/L$ is the square of the Brunt–Väisälä frequency, $\epsilon = U/\Omega L$ is the Rossby number, and

$$\delta = \alpha g \Delta T/\Omega U.$$

The parameter $\delta$ involves a relation between the value of the characteristic velocity $U$ and the characteristic perturbation temperature $\Delta T$ and is usually chosen to achieve certain balances in the equations when the flow is driven by either a given $U$ or a given $\Delta T$. In general, for an initial-value problem, both $U$ and $\Delta T$ are characterized by the initial conditions. However, to recover the flow of a homogeneous fluid in the limit $S \to 0$, balance requirements, discussed in § 4, indicate that an appropriate scaling for $\delta$ is $\delta = O(S^{\frac{1}{2}})$, which implies that

$$\Delta T = O(\epsilon \Delta T_0 S^{-\frac{1}{2}}) = O[U(\Delta T_0/L\alpha g)^{\frac{1}{2}}]. \tag{2.2a}$$

We will, therefore, use the relation,

$$\delta = S^{\frac{1}{2}}, \tag{2.2b}$$

where, for simplicity, we assume that the $O(1)$ proportionality constant is absorbed in either $U$ or $\Delta T$, and we will consider problems for which the characteristic dimensional initial values vary in accordance with the scaling (2.2$a$) in the limit $S \to 0$.

The equations for the linear theory of rotating, stratified, non-dissipative flow (Howard & Siegmann) are obtained from (2.1) with the assumption that the Rossby and Ekman numbers are small enough that the terms multiplied by either $\epsilon$ or $E$ can be neglected. With this approximation the equations become

$$\nabla . \mathbf{q} = 0, \tag{2.3a}$$

$$\frac{\partial \mathbf{q}}{\partial t} + 2\hat{\mathbf{k}} \times \mathbf{q} = -\nabla p + S^{\frac{1}{2}} T \hat{\mathbf{k}}, \tag{2.3b}$$

$$\frac{\partial T}{\partial t} + S^{\frac{1}{2}} \mathbf{q} . \hat{\mathbf{k}} = 0, \tag{2.3c}$$

where the equation of state has been used to eliminate $\rho$. The corresponding boundary and initial conditions, for flows in closed rigid containers, are

$$\mathbf{q} . \hat{\mathbf{n}} = 0, \quad \text{on } \Sigma \text{ (the container surface)}, \tag{2.4}$$

where $\hat{\mathbf{n}}$ is the outward pointing unit normal vector to $\Sigma$, and

$$\mathbf{q}(\mathbf{r}, 0) = \mathbf{q}_*(\mathbf{r}), \tag{2.5a}$$

$$T(\mathbf{r}, 0) = T_*(\mathbf{r}), \tag{2.5b}$$

where it is assumed that the initial velocity distribution satisfies (2.3*a*) and (2.4).

If solutions to these equations are sought as a superposition of a time-independent geostrophic component and a time-dependent component, in the form,

$$\mathbf{q} = \mathbf{q}_g(\mathbf{r}) + \mathbf{q}'(\mathbf{r}, t), \tag{2.6a}$$

$$p = p_g(\mathbf{r}) + p'(\mathbf{r}, t), \tag{2.6b}$$

$$T = T_g(\mathbf{r}) + T'(\mathbf{r}, t), \tag{2.6c}$$

we find that the steady component, which satisfies the equations,

$$\nabla . \mathbf{q}_g = 0, \tag{2.7a}$$

$$2\hat{\mathbf{k}} \times \mathbf{q}_g = -\nabla p_g + S^{\frac{1}{2}} T_g \hat{\mathbf{k}}, \tag{2.7b}$$

$$\mathbf{q}_g . \hat{\mathbf{k}} = 0, \tag{2.7c}$$

can be written as

$$\mathbf{q}_g = \tfrac{1}{2}\hat{\mathbf{k}} \times \nabla p_g, \tag{2.8a}$$

with

$$p_{gz} = S^{\frac{1}{2}} T_g, \tag{2.8b}$$

where the second subscript on $p_g$ denotes partial differentiation, and where $p_g$ must be such that (2.4) is satisfied, but is otherwise undetermined.

It has been shown by Howard & Siegmann that the geostrophic component can be determined from the initial data by the solution to an equation which results from the time-independence of the potential vorticity. In particular, (2.3) yield the result that

$$\frac{\partial}{\partial t}(\hat{\mathbf{k}} . \nabla \times \mathbf{q} + 2S^{-\frac{1}{2}} \hat{\mathbf{k}} . \nabla T) = 0, \tag{2.9}$$

and Howard & Siegmann have shown that the steady component is uniquely determined by the equation,

$$\hat{\mathbf{k}} . \nabla \times \mathbf{q}_g + 2S^{-\frac{1}{2}} T_{gz} = \hat{\mathbf{k}} . \nabla \times \mathbf{q}_* + 2S^{-\frac{1}{2}} T_{*z} = \Pi, \tag{2.10}$$

which, when written in terms of the pressure $p_g$, is

$$\nabla_H^2 p_g + 4S^{-1} p_{gzz} = 2\Pi, \tag{2.11}$$

where $\nabla_H^2 p_g = p_{gxx} + p_{gyy}$, with the boundary conditions:

$$\text{(i)} \quad T_g = T_*, \tag{2.12a}$$

on parts of the container surface that lie in planes perpendicular to $\hat{\mathbf{k}}$ (i.e. in $z$ = const. planes), hereafter referred to as horizontal 'flat' boundaries;

$$\text{(ii)} \quad C_0(\mathbf{q}_g, T_g; \Gamma) = C_0(\mathbf{q}_*, T_*; \Gamma), \tag{2.12b}$$

for all $\Gamma$, where

$$C_0(\mathbf{q}_g, T_g; \Gamma) = \oint_\Gamma (\mathbf{q}_g \times \hat{\mathbf{k}} + 2S^{-\frac{1}{2}} T_g \hat{\mathbf{k}}) \,.\, \hat{\mathbf{n}} |\hat{\mathbf{k}} \times \hat{\mathbf{n}}|^{-1} d\tilde{s},$$

and where $\Gamma$ is the closed contour, with arc length $d\tilde{s}$, formed by the intersection of a $z$ = const. plane with the non-horizontal part of the container surface;

$$\text{(iii)} \quad p_g = \text{const., on } \Gamma. \tag{2.12c}$$

We point out that, for the problem under consideration, Siegmann (1968) has proved that, in place of boundary condition (ii), the following condition may be used:

$$\text{(ii}a\text{)} \quad \iint_{A(z)} T_g \, dx \, dy = \iint_{A(z)} T_* \, dx \, dy, \tag{2.12d}$$

for all $A(z)$, where the integral is over the area $A(z)$ in a horizontal ($z$ = const.) plane enclosed by the boundaries of the container. We will later find it convenient to use (ii$a$) in place of (ii). For use with (2.11), the boundary conditions (2.12) can be written in terms of the pressure $p_g$ with the substitution of (2.8$a, b$).

The initial-value problem for the inviscid motion of a *homogeneous* fluid, the governing equations of which are (2.3$a, b$) with $S \equiv 0$ and the boundary and initial conditions of which are (2.4) and (2.5$a$), has been studied by Greenspan (1965, 1969) (see also Greenspan 1968, § 2.5–2.11). The solution can be sought as the superposition of a steady geostrophic flow and of a time-dependent component (2.6$a, b$), where, in several cases, it has been possible to represent the time-dependent component as a superposition of inertial modes. For containers, with top and bottom surfaces given respectively by $z = f(x, y)$ and $z = -g(x, y)$, where all contours of constant height $h = f + g$ (geostrophic contours) are closed curves on the container surface, the pressure $p_g$ is a function of $h$ and the geostrophic velocity can be written in the form,

$$\mathbf{q}_g = -\frac{1}{2} \frac{dp_g}{dh}(h) \, \mathbf{n}_T \times \mathbf{n}_B, \tag{2.13}$$

where $\mathbf{n}_T$ and $\mathbf{n}_B$ are outward pointing normal vectors to the top and bottom surfaces and are given by

$$\mathbf{n}_T = \hat{\mathbf{k}} - \nabla f = [1 + (\nabla f)^2]^{\frac{1}{2}} \, \hat{\mathbf{n}}_T, \tag{2.14a}$$

$$\mathbf{n}_B = -\hat{\mathbf{k}} - \nabla g = [1 + (\nabla g)^2]^{\frac{1}{2}} \, \hat{\mathbf{n}}_B, \tag{2.14b}$$

where $\hat{\mathbf{n}}_T$ and $\hat{\mathbf{n}}_B$ are unit normal vectors.

As described by Greenspan (1969), the mean circulation of the geostrophic flow, around a geostrophic contour $C(h)$, equals that of the initial velocity field, that is,

$$MC(\mathbf{q}_g;h) = MC(\mathbf{q}_*;h), \tag{2.15}$$

where we use the notation

$$MC(\mathbf{q}_g;h) = \oint_{C(h)} \langle \mathbf{q}_g \rangle . d\mathbf{s} \tag{2.16a}$$

and

$$\langle \mathbf{q}_g \rangle = \int_{-g}^{f} \mathbf{q}_g dz, \tag{2.16b}$$

where the directed arc length along $C$ is

$$d\mathbf{s} = ds\mathbf{n}_T \times \mathbf{n}_B / |\mathbf{n}_T \times \mathbf{n}_B|.$$

The $z$-average in (2.16$b$) will also be referred to as the vertical average.

In addition to (2.15), the mean circulation of the time-dependent component is zero, i.e.

$$MC(\mathbf{q}';h) = 0, \tag{2.17}$$

and, in terms of the initial data, the geostrophic velocity (2.13) can be expressed as

$$\mathbf{q}_g = \frac{MC(\mathbf{q}_*;h)}{J(h)} \mathbf{n}_T \times \mathbf{n}_B, \tag{2.18}$$

where

$$J(h) = h \oint_{C(h)} \mathbf{n}_T \times \mathbf{n}_B . d\mathbf{s} = h \oint_{C(h)} |\mathbf{n}_T \times \mathbf{n}_B| \, ds. \tag{2.19}$$

When the geostrophic contours are not uniquely defined, i.e. when $h$ is a constant and $\mathbf{n}_T = -\mathbf{n}_B$ (Greenspan 1968, § 2.11), the geostrophic velocity can be written as

$$\mathbf{q}_g = \tfrac{1}{2}\mathbf{n}_T \times \nabla p_g, \tag{2.20}$$

and condition (2.15) is replaced by

$$h\mathbf{n}_T . \nabla \times \mathbf{q}_g = \mathbf{n}_T . \nabla \times \langle \mathbf{q}_* \rangle. \tag{2.21}$$

The geostrophic pressure is determined by the solution to (2.21), after the substitution of (2.20), with a boundary condition derived from (2.4).

We call attention to the basically different forms of the conditions used to determine the steady geostrophic mode in a homogeneous, rotating fluid and in a stratified, rotating fluid. In the homogeneous case, the specifying equation, (2.15) or (2.21), results from the time-independence of a vertically averaged quantity which, in a container with uniquely defined geostrophic contours, is also integrated around a contour. In a stratified fluid, however, the basic equation (2.10) results from the time-independence of the potential vorticity at every spatial point. Nevertheless, if we consider solutions to the stratified initial-value problem we expect to recover, in the limit $S \to 0$, the solutions for a homogeneous fluid. We can see, however, that, according to (2.7$c$), the steady geostrophic mode will, for all values of $S$ not identically equal to zero, have $\mathbf{q}_g . \hat{\mathbf{k}} = 0$. In the limit $S \to 0$, therefore, this steady component cannot possibly approach the steady flow of a homogeneous fluid in a container whose geostrophic contours require a non-zero value of $\mathbf{q}_g . \hat{\mathbf{k}}$. This fact obscures the exact relationship of the limiting flow to the flow of a homogeneous fluid. As was pointed out by Howard & Siegmann, to relate the two cases it is necessary to consider slow unsteady motions with time scales dependent on $S$. This problem is considered in the following sections.

## 3. The inviscid solution in the limit $S \to 0$: 'flat' geostrophic contours

In § 3 we consider the relationship, in containers with 'flat' contours of constant height, of the limiting flow of a stratified fluid, as $S \to 0$, to the flow of a homogeneous fluid. Typical examples of containers with 'flat' contours are a spherical shell and a right circular cylinder with its axis aligned with the rotation vector. In the spherical shell, of course, the geostrophic contours are uniquely defined, whereas in the cylinder they are not. We will also assume, for all of the containers considered in this paper, that the geostrophic contours are closed. This eliminates any consideration of geometries, such as the 'sliced' cylinder, the flow in which was studied, for a homogeneous fluid, by Pedlosky & Greenspan (1967).

For containers with 'flat' contours, the geostrophic flow of a homogeneous fluid has no component of velocity in the vertical direction (i.e. parallel to $\hat{\mathbf{k}}$). In this case it can be shown that the steady mode of a homogeneous fluid is included in the steady geostrophic flow of a stratified fluid and that, with the exception of boundary-layer solutions in regions of zero volume, it is the limiting solution of (2.7 $a, b, c$) determined by (2.10) and boundary conditions (2.12 $a, c, d$).

First, considering a container with a uniquely defined set of contours, we can show from (2.10) and the boundary conditions (2.12 $a, c, d$) that the mean circulation of the geostrophic component of the flow of a stratified fluid is equal to that of the initial flow. This is accomplished by integrating (2.10) over an area in a horizontal plane bounded by any geostrophic contour and then over the vertical distance $h$ between bounding top and bottom contours. With the subsequent use of boundary condition (2.12 $d$) and Stokes' theorem, it follows directly that

$$MC(\mathbf{q}_g; h) = MC(\mathbf{q}_*; h). \tag{3.1}$$

When the top and bottom surfaces are 'flat' and the geostrophic contours are not unique, the integration of (2.10) over the height $h$ and the use of boundary condition (2.12 $a$) yields

$$\hat{\mathbf{k}} . \nabla \times \langle \mathbf{q}_g \rangle = \hat{\mathbf{k}} . \nabla \times \langle \mathbf{q}_* \rangle. \tag{3.2}$$

We next consider, in the limit $S \to 0$, the particular solution to (2.7 $a, b, c$) determined by (2.10) and (2.12 $a, c, d$). The complete solution evidently has the following expansion:

$$\mathbf{q} = \mathbf{q}_{g0}(\mathbf{r}) + \mathbf{q}_0(\mathbf{r}, t) + S^{\frac{1}{2}}(\mathbf{q}_{g1}(\mathbf{r}) + \mathbf{q}_1(\mathbf{r}, t)) + \ldots, \tag{3.3a}$$

$$p = p_{g0}(\mathbf{r}) + p_0(\mathbf{r}, t) + S^{\frac{1}{2}}(p_{g1}(\mathbf{r}) + p_1(\mathbf{r}, t)) + \ldots, \tag{3.3b}$$

$$T = T_{g0}(\mathbf{r}) + T_0(\mathbf{r}, t) + S^{\frac{1}{2}}(T_{g1}(\mathbf{r}) + T_1(\mathbf{r}, t)) + \ldots. \tag{3.3c}$$

Substituting this expansion in (2.3) and solving for $\mathbf{q}_{g0}$, we obtain

$$\mathbf{q}_{g0} = \tfrac{1}{2}\hat{\mathbf{k}} \times \nabla p_{g0}, \tag{3.4}$$

where

$$\hat{\mathbf{k}} . \nabla \mathbf{q}_{g0} = \hat{\mathbf{k}} . \nabla p_{g0} = 0. \tag{3.5}$$

It follows that $p_{g0} = p_{g0}(x, y)$, and, therefore, for containers with unique contours, it is implied by (2.12*c*) that

$$p_{g0} = p_{g0}(h). \tag{3.6}$$

We also find, upon substituting (3.3) into (3.1) that

$$MC(\mathbf{q}_{g0}; h) = MC(\mathbf{q}_*; h). \tag{3.7}$$

For containers with non-unique contours, we obtain, in a similar manner, from (3.2) that

$$h\hat{\mathbf{k}}.\nabla\times\mathbf{q}_{g0} = \hat{\mathbf{k}}.\nabla\times\langle\mathbf{q}_*\rangle. \tag{3.8}$$

Equations (3.4), (3.5) and (3.6) show that the lowest-order steady flow is a possible geostrophic flow of a homogeneous fluid, and (3.7) and (3.8) show that, depending on the geometry of the container, it has either the same value of the mean circulation or the same value of the $z$-averaged vertical component of vorticity as the initial velocity field. Therefore, for containers with 'flat' contours of constant height the steady geostrophic component of a stratified fluid contains the geostrophic mode of a homogeneous fluid and reduces to it in the limit $S \to 0$.

These considerations, however, do not include the possibility of boundary-layer solutions in regions of zero volume in the limit $S \to 0$. These, in fact, can occur and to demonstrate this and to identify other points of interest concerning the limiting flow we consider the initial-value problem for a weakly stratified fluid.

Assuming that the complete solution has an expansion that proceeds as (3.3), we obtain, upon substituting this expansion into (2.3), the following sets of equations:

$$\nabla.\mathbf{q}_{g0} = 0, \tag{3.9a}$$

$$2\hat{\mathbf{k}}\times\mathbf{q}_{g0} = -\nabla p_{g0}; \tag{3.9b}$$

$$\nabla.\mathbf{q}_0 = 0, \tag{3.10a}$$

$$\frac{\partial}{\partial t}\mathbf{q}_0 + 2\hat{\mathbf{k}}\times\mathbf{q}_0 = -\nabla p_0, \tag{3.10b}$$

$$\frac{\partial T_0}{\partial t} = 0; \tag{3.10c}$$

$$\nabla.\mathbf{q}_{g1} = 0. \tag{3.11a}$$

$$2\hat{\mathbf{k}}\times\mathbf{q}_{g1} = -\nabla p_{g1} + T_{g0}\hat{\mathbf{k}}, \tag{3.11b}$$

where the boundary condition (2.4) must be satisfied by each velocity vector in (3.3*a*).

### 3.1. *Cylindrical container*

We will choose a specific container shape and first consider the initial-value problem for the flow in a right circular cylinder with axis aligned with the rotation vector. In cylindrical polar co-ordinates the container surface is given by $r = r_0$, $z = 0$ and $z = 1$.

The lowest-order equations (3.9) and (3.10) are the same as those for a homogeneous fluid, and they imply (3.5). The geostrophic velocity can be expressed as given in (3.4) and it follows, in the same manner as for a homogeneous fluid, that the geostrophic flow is determined by (3.8).

The time-dependent component, satisfying (3.10), can be sought in the form of a superposition of oscillatory modes, and, to the first approximation and with the exception of the modes with very low frequencies of $O(S^{\frac{1}{2}})$,† these are the same as those of a homogeneous fluid. To higher order, there will, of course, be frequency corrections depending on $S$ and the unsteady terms in the expansion (3.3) should properly be written $\mathbf{q}_0 = \mathbf{q}_0(\mathbf{r}, t, S^{\frac{1}{2}}t, St, \ldots)$, etc., where this notation reflects the use of the method of multiple time scales (see, for example, Cole 1968, ch. 3), which will be employed in several instances in what follows. The determination of these corrections to the basic inertial modes of a homogeneous fluid is, in principle, straightforward and will not be discussed for any of the problems considered here.

Continuing with the determination of the first-order steady velocity field $\mathbf{q}_{g1}$, we tentatively, as a result of (3.10*c*), set $T_0 = 0$, and consider that the lowest-order temperature is completely represented by $T_{g0}(\mathbf{r})$. It then follows that $T_{g0}(\mathbf{r}) = T_*(\mathbf{r})$ and the solution for $\mathbf{q}_{g1}$, which satisfies (3.11), can be written in the form,

$$\mathbf{q}_{g1} = \tfrac{1}{2}\hat{\mathbf{k}} \times \nabla \left[ \int_0^z T_{g0}\, dz + \phi(x, y) \right], \tag{3.12}$$

where the function $\phi(x,y)$ is to be determined. The boundary condition (2.4) is satisfied on the top and bottom surface, but we find that (3.12) cannot possibly satisfy (2.4) at the side wall unless $T_{*\theta}(r = r_0) = 0$. This would appear to indicate that some additional unsteady motion must absorb the non-axisymmetric part of the initial temperature distribution at the cylinder side wall. A hint of this can be seen by considering the equation, obtained from (2.3), for the radial component of vorticity,

$$\frac{\partial}{\partial t}\hat{\mathbf{r}} \,.\, \nabla \times \mathbf{q} - 2\hat{\mathbf{k}} \,.\, \nabla(\mathbf{q} \,.\, \hat{\mathbf{r}}) = \frac{S^{\frac{1}{2}}}{r}\frac{\partial T}{\partial \theta},$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. If this equation is evaluated at $r = r_0$, where $\mathbf{q} \,.\, \hat{\mathbf{r}} = 0$, we can see that a non-zero value of $T_\theta(r = r_0)$ could drive an unsteady motion on a long (compared with $t$) time scale $\tau = S^{\frac{1}{2}}t$. Further investigation verifies this and shows that this motion is limited to a thin boundary layer with thickness $O(S^{\frac{1}{2}})$ at $r = r_0$.

To investigate the behaviour of the flow in this layer, therefore, we define the stretched variables,

$$\rho = S^{-\frac{1}{2}}(r_0 - r), \quad \tau = S^{\frac{1}{2}}t,$$

and, using cylindrical polar co-ordinates, scale the boundary-layer correction variables, which are all functions of $(\rho, \theta, z, \tau)$, as

$$u = S^{\frac{1}{2}}\bar{u}_1 + \ldots, \quad w = \bar{w}_0 + \ldots, \quad T = \bar{T}_0 + \ldots,$$
$$v = \bar{v}_0 + \ldots, \quad p = S^{\frac{1}{2}}\bar{p}_1 + \ldots,$$

where $u$, $v$ and $w$ are the velocity components in the $\rho$, $\theta$ and $z$ directions, respectively.

† A separate analysis, which we omit, is needed for the modes with frequencies of $O(S^{\frac{1}{2}})$. We mention, however, that it has been shown by Siegmann (1968) that, in the cylinder, the modes that are similar in structure to those in a homogeneous fluid have frequencies $\lambda^2 > S$.

The resulting boundary-layer equations are

$$\left.\begin{aligned} &-\bar{u}_{1\rho}+r_0^{-1}\bar{v}_{0\theta}+\bar{w}_{0z}=0, \quad -2\bar{v}_0-\bar{p}_{1\rho}=0,\\ &\bar{v}_{0\tau}+2\bar{u}_1+r_0^{-1}\bar{p}_{1\theta}=0, \quad \bar{w}_{0\tau}+\bar{p}_{1z}-\bar{T}_0=0,\\ &\bar{T}_{0\tau}+\bar{w}_0=0. \end{aligned}\right\} \tag{3.13}$$

We note that, while most of the original terms remain, the lowest-order balance in the radial direction is quasi-geostrophic.

The solutions to (3.13) have both a steady and a time-dependent component. The steady component, which we denote by a subscript $g$, adjusts the velocity field $\mathbf{q}_{g1}$, (3.12), so that the boundary condition (2.4) is satisfied at the side wall, i.e. so that

$$\bar{u}_{g1}(\rho=0)+\hat{\mathbf{r}}\,.\,\mathbf{q}_{g1}(r=r_0)=0. \tag{3.14}$$

To fully determine the steady solution, the following relations, which are readily derivable from (3.13) and (2.4), are needed:

$$\left.\begin{aligned} &\frac{\partial}{\partial\tau}\left(-\frac{\partial\bar{v}_0}{\partial\rho}+2\frac{\partial\bar{T}_0}{\partial z}\right)=0, \quad \frac{\partial\bar{T}_0}{\partial\tau}(z=0,1)=0,\\ &\frac{\partial}{\partial\tau}\int_0^{2\pi}\bar{v}_0(\rho=0)\,d\theta=0. \end{aligned}\right\} \tag{3.15}$$

The time integration of (3.15) and an application of the initial conditions (2.5) to the combined boundary-layer and interior variables, gives, in terms of the pressure, the equations:

$$\bar{p}_{g1\rho\rho}+4\bar{p}_{g1zz}=0, \tag{3.16}$$

$$\bar{p}_{g1z}(z=0,1)=0, \tag{3.17a}$$

$$\int_0^{2\pi}\bar{p}_{g1\rho}(\rho=0)\,d\theta=-2\int_0^{2\pi}\hat{\boldsymbol{\theta}}\,.\,(\mathbf{q}_*(r=r_0)-\mathbf{q}_{g0}(r=r_0))\,d\theta=-2C_{01}, \tag{3.17b}$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the $\theta$ direction. In addition, it is required by (3.14) that

$$\frac{\partial}{\partial\theta}(\bar{p}_{g1}(\rho=0)+p_{g1}(r=r_0))=0. \tag{3.17c}$$

and by the boundary-layer nature of the solution that $\bar{p}_{g1}(\rho\to\infty)\to 0$. Equation (3.16) and boundary conditions (3.17) determine the steady component and correspond to the appropriate limiting form of the equations, (2.11) and (2.12*a*, *b*, *c*), derived by Howard & Siegmann. It is of interest to see that these relations appear explicitly in the determination of the steady mode in this side-wall boundary layer.

The resulting solution for $\bar{p}_{g1}$ is

$$\bar{p}_{g1}=\sum_{m=1}^{\infty}(A_{0m}+A_{1m}(\theta))\cos(m\pi z)\,e^{-(2m\pi)\rho},$$

where
$$A_{0m}=\frac{1}{m\pi^2}\int_0^1\cos(m\pi z)\,C_{01}\,dz,$$

and
$$A_{1m}(\theta)=\frac{2}{m\pi}\int_0^1\sin(m\pi z)\left(T_*(r=r_0)-\frac{1}{2\pi}\int_0^{2\pi}T_*(r=r_0)\,d\theta\right)dz.$$

The steady component has an $O(1)$ velocity component in the $\theta$ direction, $\bar{v}_0$, which, in the limit $s \to 0$, can remain non-zero only on the surface $r = r_0$.

For the time-dependent component, we seek the solution in the form of oscillatory modes, where we write, for example,

$$\bar{p}_1 = \hat{p}_{1n}(\rho, \theta, z)\, e^{i\lambda_n \tau}.$$

Equations (3.13) can be combined into a single equation for $\hat{p}_{1n}$, which is (dropping the subscript $n$)

$$\hat{p}_{1\rho\rho} + 4(1-\lambda^2)^{-1}\hat{p}_{1zz} = 0.$$

The boundary conditions on $\hat{p}_1$, resulting from the requirements that $\hat{u}_1(\rho = 0) = 0$ and that $\hat{w}_0(z = 0) = \hat{w}_0(z = 1) = 0$, are

$$i\lambda\hat{p}_{1\rho} - 2r_0^{-1}\hat{p}_{1\theta} = 0, \quad \text{at} \quad \rho = 0,$$

$$\hat{p}_{1z} = 0, \quad \text{at} \quad z = 0, 1.$$

In addition, we require that $\hat{p}_1 \to 0$ as $\rho \to \infty$.

The eigenfunction solutions for $\hat{p}_1$ are

$$\hat{p}_{1mk} = \cos(m\pi z)\, e^{ik\theta}\, e^{-\alpha_{mk}\rho},$$

where $m, k = \pm 1, \pm 2, \ldots$, and where

$$\alpha_{mk} = 2|m|\,\pi(1-\lambda_{mk}^2)^{-\frac{1}{2}},$$

and

$$\lambda_{mk} = -k[(m\pi r_0)^2 + k^2]^{-\frac{1}{2}}.$$

In particular, the resulting solution for $\bar{T}_0$ is

$$\bar{T}_0 = -\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}[A_{mk}\cos(k\theta + \lambda_{mk}\tau) + B_{mk}\sin(k\theta + \lambda_{mk}\tau)] \times (1-\lambda_{mk}^2)^{-1} m\pi \sin(m\pi z)\, e^{-\alpha_{mk}\rho}.$$

Note that, since $k$ and $\lambda_{mk}$ have the opposite sign, these solutions represent waves travelling strictly in the positive $\theta$ direction, i.e. in the same direction as the basic rotation. In addition, their structure in the radial direction is characterized by a simple exponential decay from the boundary. We also note that the radial velocity $\hat{u}_1 \equiv 0$.

The non-axisymmetric part of the initial temperature distribution on the cylinder side wall evidently excites this set of boundary-layer modes and the coefficients $A_{mk}$ and $B_{mk}$ can be determined, using the properties of the trigonometric functions, by the requirement that

$$\bar{T}_0(\rho = 0, \tau = 0) = T_*(r = r_0) - \frac{1}{2\pi}\int_0^{2\pi} T_*(r = r_0)\, d\theta. \tag{3.18}$$

In the limit $S \to 0$, the frequency and the extent of the region of existence of these boundary-layer oscillatory modes approach zero. In a purely homogeneous fluid they do not exist. We note that these modes are the asymptotic form of the class II modal solutions, found by Siegmann (1968), for a fluid, with an arbitrary value of $S$, in a cylindrical container. We also point out that Siegmann (1968) has considered, in general, the initial-value problem for all the modes in the cylinder and that the appropriate limiting form of the coefficients shows that, in addition to the condition given by (3.18), the class II modes can be excited independently by certain parts of the initial velocity distribution. Our asymptotic analysis gives some insight, however, into one reason that these modes arise when a weak stratification is present and also, through the boundary-layer equations (3.13), into the balance of forces involved when $S \ll 1$.

### 3.2. *Spherical container*

The situation is different in a container without vertical side walls, and we next consider the motion in a spherical container with surface specified by (retaining cylindrical polar co-ordinates) $z^2+r^2 = 1$. We assume initially the same form of expansion as before (3.3), and, of course, we find the same set of equations (3.9)–(3.11). In this case, however, the lowest-order steady component of the velocity field $\mathbf{q}_{g0}$ is determined by the mean circulation theorem (2.18). If we again tentatively assume that $T_{g0}(\mathbf{r}) = T_*(\mathbf{r})$, we find that the first-order steady velocity field $\mathbf{q}_{g1}$ cannot satisfy the boundary condition (2.4) unless $\langle T_* \rangle_\theta = 0$. We are, therefore, led again to suspect that an additional unsteady motion must arise to absorb the non-axisymmetric part of the $z$-averaged initial temperature distribution. From our knowledge of the flow in the cylindrical container, we might expect that these motions vary on a long time scale, dependent on the stratification parameter. In addition, because of the $z$-average in the above condition, we should expect the lowest-order solution for these modes to have $z$-independent values for both the temperature and the vertical velocity component and, consequently, a linear $z$ dependence for the other variables.

Further examination shows that the basic time scale of these motions is $\tau_1 = St$, and that the expansion of the correction variables proceeds as

$$\mathbf{q} = S^{\frac{1}{2}}(\tilde{\mathbf{q}}_0+S\tilde{\mathbf{q}}_1+\ldots), \tag{3.19a}$$

$$p = S^{\frac{1}{2}}(\tilde{p}_0+S\tilde{p}_1+\ldots), \tag{3.19b}$$

$$T = \tilde{T}_0+S\tilde{T}_1+\ldots, \tag{3.19c}$$

where all variables are functions of $(\mathbf{r}, \tau_1, \tau_2, \ldots)$ and where $\tau_1 = St$, $\tau_2 = S\tau_1$, etc.

The lowest-order equations are

$$\nabla . \tilde{\mathbf{q}}_0 = 0, \tag{3.20a}$$

$$2\hat{\mathbf{k}} \times \tilde{\mathbf{q}}_0 = -\nabla\tilde{p}_0+\tilde{T}_0\hat{\mathbf{k}}, \tag{3.20b}$$

$$\frac{\partial \tilde{T}_0}{\partial \tau_1}+\tilde{\mathbf{q}}_0.\hat{\mathbf{k}} = 0, \tag{3.20c}$$

with those for the next order being

$$\nabla . \tilde{\mathbf{q}}_1 = 0, \tag{3.21a}$$

$$\frac{\partial \tilde{\mathbf{q}}_0}{\partial \tau_1}+2\hat{\mathbf{k}} \times \tilde{\mathbf{q}}_1 = -\nabla\tilde{p}_1+\tilde{T}_1\hat{\mathbf{k}}, \tag{3.21b}$$

$$\frac{\partial \tilde{T}_1}{\partial \tau_1}+\tilde{\mathbf{q}}_1.\hat{\mathbf{k}} = -\frac{\partial \tilde{T}_0}{\partial \tau_2}. \tag{3.21c}$$

It follows from (3.20$a,b$) that $\hat{\mathbf{k}}.\nabla(\tilde{\mathbf{q}}_0.\hat{\mathbf{k}}) = 0$ and, consequently, from (3.20$c$) that $\partial(\tilde{T}_{0z})/\partial\tau_1 = 0$. The boundary condition (2.4) is, in this case,

$$z\mathbf{q}.\hat{\mathbf{k}}+r\mathbf{q}.\hat{\mathbf{r}} = 0 \quad \text{on} \quad z^2+r^2 = 1. \tag{3.22}$$

By substituting for the individual velocity components in (3.21*a*) we obtain

$$\frac{\partial}{\partial \tau_1}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{p}_0}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \tilde{p}_0}{\partial \theta^2}+4\frac{\partial^2 \tilde{p}_1}{\partial z^2}\right] = 0. \tag{3.23}$$

In addition, the application of boundary condition (3.22) to $\tilde{\mathbf{q}}_0$ yields, in terms of $\tilde{p}_0$,

$$2z\frac{\partial^2 \tilde{p}_0}{\partial \tau_1 \partial z}+\frac{\partial \tilde{p}_0}{\partial \theta} = 0 \quad \text{on} \quad z^2+r^2 = 1. \tag{3.24}$$

Since (3.23) contains both $\tilde{p}_0$ and $\tilde{p}_1$, an additional relation between these two variables is needed. This is obtained by applying boundary condition (3.22) to $\tilde{\mathbf{q}}_1$, and is

$$z\frac{\partial}{\partial z}\left(\frac{\partial^3 \tilde{p}_0}{\partial \tau_1^3}-\frac{\partial \tilde{p}_1}{\partial \tau_1}-\frac{\partial \tilde{p}_0}{\partial \tau_2}\right)-\tfrac{1}{2}r\left(\frac{1}{r}\frac{\partial \tilde{p}_1}{\partial \theta}+\frac{1}{2}\frac{\partial^2 \tilde{p}_0}{\partial \tau_1 \partial r}\right) = 0, \quad \text{on} \quad z^2+r^2 = 1. \tag{3.25}$$

Seeking solutions to (3.23)–(3.25) of the form

$$\tilde{p}_0 = zf_{0k}(r)\exp\left[i(k\theta+\lambda\tau_1)\right],$$

and

$$\tilde{p}_1 = \left[zg_{0k}(r)+z^3g_{1k}(r)\right]\exp\left[i(k\theta+\lambda\tau_1)\right],$$

where $k$ is an integer and where $\lambda\tau_1 = \lambda_{0k}\tau_1+\lambda_{1k}(S\tau_1)+\ldots$, we find, from (3.24), that

$$\lambda_{0k} = -\tfrac{1}{2}k, \tag{3.26}$$

which shows, in particular, that the frequency $\lambda_{0k}$ is equal to zero for the axisymmetric, $k = 0$, component of $\tilde{p}_0$ (and, therefore, for the axisymmetric component of $\tilde{T}_0$). From (3.23) and (3.24) we obtain (suppressing the $k$ subscript), respectively,

$$r(rf_0')'-k^2f_0+24r^2g_1 = 0, \tag{3.27}$$

and

$$(k^3f_0-8\lambda_1 f_0+krf_0')+8k(1-r^2)g_1 = 0, \tag{3.28}$$

where the prime denotes total differentiation. Eliminating $g_1$ from (3.27) and (3.28), we find

$$r^2(1-r^2)f_0''+r(1-4r^2)f_0'+\left[-k^2(1+2r^2)+24r^2\lambda_1 k^{-1}\right]f_0 = 0. \tag{3.29}$$

With the transformations $\xi = (1-r^2)^{\frac{1}{2}}$ and $h_0(\xi) = \xi f_0$, (3.29) reduces to

$$(1-\xi^2)h_0''-2\xi h_0'+\left[2-3k^2+24\lambda_1 k^{-1}-\xi^2(1-\xi^2)^{-1}k^2\right]h_0 = 0. \tag{3.30}$$

The appropriate boundary conditions for (3.30) result from the requirements, (3.22), that $\hat{\mathbf{k}}.\mathbf{q}(r = 0) = 0$ and $\hat{\mathbf{r}}.\mathbf{q}(r = 1) = 0$ and are

$$h_0(\xi = 0) = 0, \quad h_0(\xi = 1) = 0. \tag{3.31a, b}$$

A choice for $\lambda_1$ that allows the boundary and initial conditions to be satisfied is

$$\lambda_{1nk} = -\tfrac{1}{12}k\left[1-k^2-\tfrac{1}{2}n(n+1)\right], \tag{3.32}$$

where $n$ is a positive integer. Equation (3.30) then becomes

$$(1-\xi^2)h_0''-2\xi h_0'+\left[n(n+1)-k^2(1-\xi^2)^{-1}\right]h_0 = 0, \tag{3.33}$$

and the solutions are associated Legendre functions

$$h_{0nk} = P_n^k(\xi),$$

where, to satisfy (3.31), $k \neq 0$ and $|n-k|$ must be odd.

The resulting solution for $\tilde{p}_0$ can be written as

$$\tilde{p}_0 = z\xi^{-1} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathrm{e}_{nk} P_n^k(\xi) \{A_{nk} \cos [k(\theta - \tfrac{1}{2}\tau_1)] + B_{nk} \sin [k(\theta - \tfrac{1}{2}\tau_1)]\} \quad (3.34)$$

where

$$\mathrm{e}_{nk} = \begin{cases} 0 & \text{if} \quad |n-k| \text{ is even,} \\ 1 & \text{if} \quad |n-k| \text{ is odd,} \end{cases}$$

and where only the lowest-order approximation (3.26) for the frequency has been retained.

For the initial-value problem, the coefficients $A_{nk}$ and $B_{nk}$ can be determined from the requirement that these modes absorb the non-axisymmetric part of the $z$-averaged initial temperature distribution, i.e. that

$$\tilde{p}_{0z}(\mathbf{r}, \tau_1 = 0) = \tilde{T}_0(\mathbf{r}, \tau_1 = 0) = h^{-1}\langle T_* \rangle - \frac{h^{-1}}{2\pi} \int_0^{2\pi} \langle T_* \rangle \, d\theta. \quad (3.35)$$

The orthogonality and completeness properties of the associated Legendre functions and the trigonometric functions are naturally utilized in this step. The restriction of $|n-k|$ to odd values is acceptable since the factor $\xi^{-1}$ in (3.34) results in a multiplication of the temperature distribution by $\xi$, causing the product to vanish at $\xi = 0$ and allowing it to be extended, in the interval $-1 \leqslant \xi \leqslant 0$, as an odd function, in which case, only odd values of $|n-k|$ are required.

The lowest-order steady temperature field is then given by

$$T_{g0}(\mathbf{r}) = T_*(\mathbf{r}) - \tilde{T}_0(\mathbf{r}, \tau_1 = 0),$$

and, since $\langle T_{g0} \rangle_\theta = 0$, the solution for the first-order velocity field $\mathbf{q}_{g1}$ can be consistently determined.

These low-frequency oscillatory modes are, as expected, the asymptotic form of the class II modal solutions, found by Siegmann (1968), for a fluid, with arbitrary values of $S$, in a spherical container. We note that, as was the case with the class II modes in the cylinder, these solutions represent waves travelling in the positive $\theta$ direction and that they also absorb a non-axisymmetric part of the initial temperature distribution. Unlike the modes in the cylinder, however, the temperature, in the spherical container, is, to lowest order, $z$-independent and the modes exist throughout the fluid, not just in a wall-boundary layer. Also, the frequency is proportional to $S$, whereas in the cylinder it was proportional to $S^{\frac{1}{2}}$. An additional noteworthy feature of the modes in the sphere, as seen from (3.34), is that, to lowest order, all of the phases of the waves travel in the positive $\theta$ direction with the same angular velocity, $d\theta/d\tau_1 = \frac{1}{2}$. The frequency and the amplitudes of the velocity components both approach zero in the limit $S \to 0$ and these modes do not exist in a homogeneous fluid. Again, the asymptotic analysis clarifies, through the initial condition requirement (3.35) and through the approximate governing equations (3.20) and (3.21) and their solution (3.34), the role that these modes play in the presence of a weak stratification.

## 4. The limiting solution: 'sloping' geostrophic contours

For containers with 'sloping' geostrophic contours there is an additional basic difference in the behaviour of the limiting flow. In this situation, it is found that the lowest-order solution for the geostrophic flow must itself be time-dependent, varying in a long time scale which depends on the stratification parameter. The way in which this comes about can be readily seen, for example, in the case where the geostrophic contours are not uniquely defined.

Consider a container formed from a general cylinder, with generators parallel to the $z$-axis and with top and bottom surfaces such that

$$h = f + g = 1, \tag{4.1a}$$

and

$$\mathbf{n}_T = -\mathbf{n}_B. \tag{4.1b}$$

The 'doubly sliced' cylinder (figure 1) is a particular example of this type of geometry. In this case, the dot product of $\mathbf{n}_T$ with the vertical average of the curl of (2.3$b$) and the application of the boundary condition (2.4) yields

$$\frac{\partial}{\partial t}\mathbf{n}_T.\langle\nabla\times\mathbf{q}\rangle = -S^{\frac{1}{2}}\mathbf{n}_T\times\hat{\mathbf{k}}.\nabla\langle T\rangle, \tag{4.2}$$

where $\mathbf{n}_T\times\hat{\mathbf{k}}.\langle\nabla T\rangle = \mathbf{n}_T\times\hat{\mathbf{k}}.\nabla\langle T\rangle$. If $S = 0$ (or if $\mathbf{n}_T\times\hat{\mathbf{k}} = 0$) (4.2) gives directly the time independence of the vertically averaged component of vorticity in the $\mathbf{n}_T$ direction, which is, therefore, carried by the steady mode. Otherwise, however, (4.2) states that this $z$-averaged component of vorticity changes in time due to certain horizontal gradients in the vertically averaged temperature. The substitution of the gradient of the vertical average of (2.3$c$) into the time derivative of (4.2) eliminates the temperature and results in the equation,

$$\frac{\partial^2}{\partial t^2}\mathbf{n}_T.\langle\nabla\times\mathbf{q}\rangle = S\mathbf{n}_T\times\hat{\mathbf{k}}.\nabla\langle\mathbf{q}.\hat{\mathbf{k}}\rangle. \tag{4.3}$$

We can see from (4.3) that, with $\mathbf{n}_T\times\hat{\mathbf{k}} \neq 0$ and with $S \ll 1$, the quantity $\mathbf{n}_T.\langle\nabla\times\mathbf{q}\rangle$ $(= \mathbf{n}_T.\nabla\times\langle\mathbf{q}\rangle)$ varies on a long time scale $\tau = S^{\frac{1}{2}}t$. In the limit $S \to 0$ this unsteady motion approaches a steady flow.

With regard to our original non-dimensionalization in § 2, we point out that the time scale $\tau = S^{\frac{1}{2}}t$ is independent of the value of the parameter $\delta$. Once the scaling for $\tau$ is established, our choice for the scaling of $\delta$, (2.2), follows from the requirements that, in the limit $S \to 0$, $\mathbf{q}_{g0}.\hat{\mathbf{k}} = O(1)$ and also that $\mathbf{q}_{g0}.\hat{\mathbf{k}}$ balance the time derivative of the lowest-order temperature in (2.3$c$).

### 4.1. *Uniquely defined contours*

Let us first consider the initial-value problem for a container whose surface is composed of uniquely defined geostrophic contours. The initial procedure is similar to that of the last section with the exception that, based on the considerations just given, we must allow the geostrophic flow to vary on the $\tau$ time scale. Therefore, assuming an expansion similar to (3.3), where, however, all quantities are now assumed to depend also on $\tau$, for example,

$$\mathbf{q}_{g0} = \mathbf{q}_{g0}(\mathbf{r},\tau), \quad \mathbf{q}_0 = \mathbf{q}(\mathbf{r},t,\tau,\ldots), \quad \text{etc.,} \tag{4.4}$$

we find identical lowest-order equations (3.9) and (3.10). The first-order equations are

$$\nabla \cdot \mathbf{q}_{g1} = 0, \tag{4.5a}$$

$$\mathbf{q}_{g0\tau} + 2\hat{\mathbf{k}} \times \mathbf{q}_{g1} = -\nabla p_{g1} + T_{g0}\hat{\mathbf{k}}, \tag{4.5b}$$

$$T_{g0\tau} + \mathbf{q}_{g0} \cdot \hat{\mathbf{k}} = 0, \tag{4.5c}$$

and

$$\nabla \cdot \mathbf{q}_1 = 0, \tag{4.6a}$$

$$\mathbf{q}_{1t} + 2\hat{\mathbf{k}} \times \mathbf{q}_1 = -\nabla p_1 + T_0\hat{\mathbf{k}} - \mathbf{q}_{0\tau}, \tag{4.6b}$$

$$T_{1t} + \mathbf{q}_0 \cdot \hat{\mathbf{k}} = -T_{0\tau}, \tag{4.6c}$$

with the boundary conditions,

$$\mathbf{q}_{g1} \cdot \hat{\mathbf{n}} = \mathbf{q}_1 \cdot \hat{\mathbf{n}} = 0, \text{ on the container surface.} \tag{4.7}$$

Since the lowest-order equations are the same as those for a homogeneous fluid, it follows that

$$\mathbf{q}_{g0} = -\frac{1}{2}\frac{\partial p_{g0}}{\partial h}(h, \tau)\, \mathbf{n}_T \times \mathbf{n}_B = f(h, \tau)\, \mathbf{n}_T \times \mathbf{n}_B, \tag{4.8}$$

and that

$$MC(\mathbf{q}_{g0}(\tau = 0); h) = MC(\mathbf{q}_*; h), \tag{4.9a}$$

with

$$MC(\mathbf{q}_0; h) = 0. \tag{4.9b}$$

To determine the function $f$ in (4.8), it is necessary to consider the first-order equations (4.5). These can be manipulated in a manner similar to that used by Greenspan (1965) in the spin-up problem. In our case, however, considerable algebraic and conceptual simplification can be achieved if it is realized that the procedure to be followed is equivalent to that of taking the mean circulation of (4.5*b*) and using the result (Howard 1968, p. 49) that, with $\nabla \cdot \mathbf{q} = 0$,

$$MC(2\hat{\mathbf{k}} \times \mathbf{q} + \nabla p; h) = -2\iint_{\Sigma'_T(h)} \mathbf{q} \cdot \hat{\mathbf{n}}_T \, d\Sigma'_T - 2\iint_{\Sigma'_B(h)} \mathbf{q} \cdot \hat{\mathbf{n}}_B \, d\Sigma'_B, \tag{4.10}$$

where $\Sigma'_T(h)$ and $\Sigma'_B(h)$ are the portions of the top and bottom surfaces enclosed by the particular geostrophic contour $C = C(h)$ on that surface.

To determine $f$, therefore, we take the mean circulation of (4.5*b*), and use (4.10) in conjunction with the boundary condition (4.7), to obtain

$$\frac{\partial}{\partial \tau} MC(\mathbf{q}_{g0}; h) = MC(T_{g0}\hat{\mathbf{k}}; h). \tag{4.11}$$

Next, taking the mean circulation of (4.5*c*) multiplied by $\hat{\mathbf{k}}$, we find

$$\frac{\partial}{\partial \tau} MC(T_{g0}\hat{\mathbf{k}}; h) = -MC[(\mathbf{q}_{g0} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}; h], \tag{4.12}$$

which, when substituted in the $\tau$ derivative of (4.11), yields

$$\frac{\partial^2}{\partial \tau^2} MC(\mathbf{q}_{g0}; h) + MC[(\mathbf{q}_{g0} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}; h] = 0. \tag{4.13}$$

Finally, substituting (4.8) into (4.13), we obtain the equation for $f$:

$$\frac{\partial^2 f}{\partial \tau^2} + \frac{K}{J} f = 0, \tag{4.14}$$

where $J = J(h)$ is given by (2.19), and where

$$K = K(h) = h \oint_{C(h)} (\mathbf{n}_T \times \mathbf{n}_B . \hat{\mathbf{k}}) \hat{\mathbf{k}} . d\mathbf{s}$$

$$= h \oint_{C(h)} \frac{|\mathbf{n}_T \times \mathbf{n}_B . \hat{\mathbf{k}}|^2}{|\mathbf{n}_T \times \mathbf{n}_B|} ds. \tag{4.15}$$

The solution to (4.14), since $K/J \geqslant 0$, is simply

$$f = C_1(h) \cos (K/J)^{\frac{1}{2}} \tau + C_2(h) \sin (K/J)^{\frac{1}{2}} \tau. \tag{4.16}$$

Before we can determine the coefficients $C_1$ and $C_2$ from the initial conditions, however, we need an additional result. If we take the mean circulation of (4.6*b*) and use (4.10), (4.7) and (4.9*b*), we find

$$\frac{\partial}{\partial t} MC(\mathbf{q}_1; h) = MC(T_0 \hat{\mathbf{k}}; h). \tag{4.17}$$

Since $T_0$ is independent of $t$ by (3.10*c*), we must require, to avoid a secular growth rate in $t$ of $MC(\mathbf{q}_1; h)$, that

$$MC(T_0 \hat{\mathbf{k}}; h) = 0. \tag{4.18}$$

It then follows from (4.11) and (4.18) that

$$MC\left(\frac{\partial}{\partial \tau} \mathbf{q}_{g0}(\tau = 0); h\right) = MC(T_{g0}(\tau = 0) \hat{\mathbf{k}}; h) = MC(T_* \hat{\mathbf{k}}; h). \tag{4.19}$$

With the use of (4.9*a*) and (4.19), the solution for $\mathbf{q}_{g0}$ can be written

$$\mathbf{q}_{g0} = [J^{-1} MC(\mathbf{q}_*; h) \cos (K/J)^{\frac{1}{2}} \tau + (JK)^{-\frac{1}{2}} MC(T_* \hat{\mathbf{k}}; h) \times \sin (K/J)^{\frac{1}{2}} \tau] \, \mathbf{n}_T \times \mathbf{n}_B, \tag{4.20a}$$

and it follows that

$$T_{g0} = [-(JK)^{-\frac{1}{2}} MC(\mathbf{q}_*; h) \sin (K/J)^{\frac{1}{2}} \tau + K^{-1} MC(T_* \hat{\mathbf{k}}; h) \times \cos (K/J)^{\frac{1}{2}} \tau] \, \mathbf{n}_T \times \mathbf{n}_B . \hat{\mathbf{k}}, \tag{4.20b}$$

where (4.20*b*) has been obtained by the substitution of (4.20*a*) into (4.5*c*) and the subsequent integration of (4.5*c*) with respect to $\tau$. The constant of integration has been ignored, and it will be assumed that the steady part of the lowest-order temperature field is written as a separate component $T_{0s}$, where, however, a consideration of (4.11) leads to the conclusion that $MC(T_{0s} \hat{\mathbf{k}}; h) = 0$. Note that for 'flat' contours we have $\hat{\mathbf{k}} . \mathbf{n}_T \times \mathbf{n}_B = 0$, which implies that $K = 0$, and, therefore, the oscillatory behaviour in (4.20*a*, *b*) is not present.

The solution for $\mathbf{q}_{g0}$ is similar in structure to that of the geostrophic flow of a homogeneous fluid, except that it has an oscillatory behaviour, with frequency dependent on $h$, on the long $\tau$ time scale. In the limit $S \to 0$, $\mathbf{q}_{g0}$ approaches the geostrophic flow of a homogeneous fluid determined by the mean circulation theorem.

Guided by the results of the last section, for containers with 'flat' contours, we should expect additional unsteady motions to arise in response to certain distributions of the initial temperature field. However, we do not attempt to describe these motions in this (or in the following) case. Also, we mention that some part of a general initial temperature distribution will presumably go into a completely steady mode $T_{0s}$ for which it appears that the amplitude of the corresponding lowest-order, steady velocity components will be $O(S^{\frac{1}{2}})$.

### 4.2. *Non-unique contours*

For containers with non-unique contours, whose description was given at the beginning of this section (4.1 *a*, *b*), the treatment of the initial-value problem proceeds with the same expansion, (3.3) and (4.4), as that just used and the resulting sets of equations (3.9), (3.10), (4.5) and (4.6) are, of course, the same. The determining equation for the lowest-order variables can be found from the first-order equations (4.5) by using a properly modified form of the procedure employed before to find (4.14). In fact, however, the appropriate relation has already been obtained in (4.3). Substituting the expansion (3.3), modified as given by (4.4), and the expression for the velocity

$$\mathbf{q}_{g0} = \tfrac{1}{2}\mathbf{n}_T \times \nabla p_{g0}, \tag{4.21}$$

which can be obtained from (3.9) and the boundary conditions (2.4), into (4.3), we find the governing equation for $p_{g0}$:

$$\frac{\partial^2}{\partial\tau^2}\mathbf{n}_T\,.\,\nabla\times(\mathbf{n}_T\times\nabla p_{g0}) = -(\mathbf{n}_T\times\hat{\mathbf{k}}\,.\,\nabla)^2 p_{g0}. \tag{4.22}$$

The boundary condition (2.4) on the vertical side wall, with unit normal vector $\hat{\mathbf{n}}_s$, is, with the use of (4.21), $\hat{\mathbf{n}}_s \times \mathbf{n}_T\,.\,\nabla p_{g0} = 0$, which, with (3.5), can be taken as

$$p_{g0} = 0, \text{ on the side wall.} \tag{4.23}$$

Since the lowest-order equations (3.9) and (3.10) are the same as those for a homogeneous fluid, it follows that

$$\mathbf{n}_T\,.\,\nabla\times\mathbf{q}_{g0}(\tau=0) = \mathbf{n}_T\,.\,\nabla\times\langle\mathbf{q}_*\rangle. \tag{4.24}$$

In addition, considering (4.5 *b*), and using arguments similar to those leading to (4.18), we obtain

$$\mathbf{n}_T\times\hat{\mathbf{k}}\,.\,\nabla\langle T_0\rangle = 0, \tag{4.25}$$

from which it follows that

$$\mathbf{n}_T\times\hat{\mathbf{k}}\,.\,\nabla T_{g0}(\tau=0) = \mathbf{n}_T\times\hat{\mathbf{k}}\,.\,\nabla\langle T_*\rangle. \tag{4.26}$$

In terms of the pressure, the initial conditions (4.24) and (4.26) can be expressed as

$$\mathbf{n}_T\,.\,\nabla\times[\mathbf{n}_T\times\nabla p_{g0}(\tau=0)] = 2\mathbf{n}_T\,.\,\nabla\times\langle\mathbf{q}_*\rangle, \tag{4.27a}$$

and

$$\mathbf{n}_T\,.\,\nabla\times\left[\mathbf{n}_T\times\nabla\frac{\partial p_{g0}}{\partial\tau}(\tau=0)\right] = -2\mathbf{n}_T\times\hat{\mathbf{k}}\,.\,\nabla\langle T_*\rangle. \tag{4.27b}$$

Equation (4.22), with boundary condition (4.23) and with initial conditions given by the solutions to (4.27 *a*, *b*) (which also have boundary condition (4.23)), determines the geostrophic pressure. The velocity is then determined by (4.21), and, with the use of (4.21), the temperature $T_{g0}$ can be obtained from the integration of (4.5 *c*) with respect to $\tau$. In the limit $S \to 0$ the time scale of these motions goes to zero and the flow approaches the geostrophic steady flow of a homogeneous fluid determined by (4.24).

Let us consider the particular case of the 'doubly sliced' cylinder (figure 1) for which $g = -(b/a)y, f = 1-g$ and

$$\mathbf{n}_T = -\mathbf{n}_B = \hat{\mathbf{k}} - (b/a)\mathbf{j}, \tag{4.28}$$

where $a$ and $b$ are the direction cosines, such that $a^2+b^2 = 1$. With the substitution of (4.28) into (4.22), we obtain

$$\frac{\partial^2}{\partial\tau^2}\nabla_T^2 p_{g0} = -b^2\frac{\partial^2 p_{g0}}{\partial x^2}, \tag{4.29}$$

where $\nabla_T^2 p_{g0} = p_{g0xx} + (1-b^2)p_{g0yy}$. If a solution to (4.29) is sought in the form,

$$p_{g0} = p_m \exp(i\sigma_m t), \tag{4.30}$$

equation (4.29) becomes

$$\frac{\partial^2 p_m}{\partial y^2} - \mu_m^2 \frac{\partial^2 p_m}{\partial x^2} = 0, \tag{4.31}$$

where

$$\sigma_m^2 = b^2(1+a^2\mu_m^2)^{-1}, \tag{4.32}$$

and where, with boundary condition (4.23), only the above choice of sign for $p_{mxx}$ leads to non-zero solutions. It follows from (4.32) that $\sigma_m^2 \leqslant b^2$, and, therefore, with account taken for the $\tau$ scaling, that all the frequencies for these modes are smaller in absolute value than $bS^{\frac{1}{2}}$ ($bN$ in dimensional units).

The solution to (4.31) with boundary condition (4.23), for a rectangular region $0 \leqslant x \leqslant x_0$, $0 \leqslant y \leqslant y_0$, has been given by Høiland (1962). The resulting expression for $p_{g0}$ is

$$p_{g0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos\sigma_{mn}t + B_{mn}\sin\sigma_{mn}t)\sin\left(\frac{m\pi x}{x_0}\right)\sin\left(\frac{n\pi y}{y_0}\right), \tag{4.33a}$$

where $\mu_{mn}^2 = (nx_0/my_0)^2$ and, therefore, where

$$\sigma_{mn} = mby_0[(nax_0)^2 + (my_0)^2]^{-\frac{1}{2}}. \tag{4.33b}$$

We note that for each eigenvalue there is a corresponding infinite number of distinct eigenfunctions. For 'flat' contours ($b \equiv 0$), the solutions to (4.31) are $p_m = 0$ and no modes of this type exist.

For the initial-value problem, the quantities $p_{g0}(\tau = 0)$ and $p_{g0\tau}(\tau = 0)$ are given by the solutions to (4.27*a, b*) with boundary condition (4.23). In the case of the rectangular cylinder, the coefficients $A_{mn}$ and $B_{mn}$ can then be determined in the standard manner by using the known completeness and orthogonality of the terms in the double sine series.

The solution to (4.31), for a circular region $0 \leqslant r \leqslant r_0$, has been studied by Barcilon (1968). Different boundary conditions were used for the pressure, but the eigenfunction solutions given there for the stream function correspond, with a modification of the eigenvalues, to the solutions for the pressure in our problem. The relation between the eigenvalues $\lambda$ in that paper and those in our case is $\mu^2 = \lambda^2(4-\lambda^2)^{-1}$ [for $(\mu; x, y) \rightarrow (\lambda; x, z)$]. We mention that again there is an infinite number of eigenfunctions for every eigenvalue. For the solution to the initial-value problem, the completeness and the appropriate orthogonality conditions of these modes would have to be considered, but we do not bother with that here.

## 5. Weak stratification waves

The modal solutions to (4.29) might be called 'contained weak stratification waves'. They are a particular class of internal-inertial waves which arise when a weak stratification restricts the lowest-order geostrophic flow. As mentioned by Howard & Siegmann the appearance of these low-frequency wave motions, with the addition of a weak stratification, is reminiscent of the appearance of low-frequency Rossby waves when a small change in geometry restricts the geostrophic flow of a homogeneous fluid (Pedlosky & Greenspan 1967; also Greenspan 1968, §2.16).

In these waves the essential mechanism involved is the interaction of the geostrophic flow and the basic stratification. This point is emphasized by the fact that, if the angle of slope of the top and bottom surfaces of the 'doubly sliced' cylinder is small (i.e. if $b \ll 1$), (4.29) can be approximated by

$$\frac{\partial^2}{\partial\tau^2}\nabla_H^2 p_{g0} = -b^2\frac{\partial^2 p_{g0}}{\partial x^2}, \tag{5.1}$$

and this is the same form of equation as would arise for a weakly stratified, rotating fluid in a container with 'flat' top and bottom surfaces and with the basic stratification (with Brunt–Väisälä frequency $bN$) in *the y direction*. We also note that (5.1) has the same form as that governing $z$-independent *internal* waves in a fluid which is stratified in the $y$ direction. In this respect, the weak stratification waves can be looked on essentially as internal waves whose structure is held $z$-independent by the constraint of the rotation.

These weak stratification waves can be expected in other situations where the elements of the basic interaction are present. For example, suppose that we have a situation where the centrifugal forces are not negligible and where, for the time scales of interest, the equilibrium surfaces of constant density are given by (see Greenspan 1968, §1.4)

$$z = \tfrac{1}{2}Fr^2 + \text{const.}, \tag{5.2}$$

where $r$ is the radial distance, in cylindrical polar co-ordinates, from the centre of rotation. In this case, (2.3$b,c$) are replaced by

$$\mathbf{q}_t + 2\hat{\mathbf{k}}\times\mathbf{q} = -\nabla p + S^{\frac{1}{2}}T(\hat{\mathbf{k}} - Fr\hat{\mathbf{r}}), \tag{5.3$a$}$$

$$T_t + S^{\frac{1}{2}}\mathbf{q}\,.\,(\hat{\mathbf{k}} - Fr\hat{\mathbf{r}}) = 0, \tag{5.3$b$}$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction.

Let us consider the motion in a circular cylinder ($0 \leqslant r \leqslant r_0$, $0 \leqslant z \leqslant 1$) with 'flat' top and bottom surfaces and with axis aligned with the rotation vector and placed at the centre of rotation. In this geometry, the basic stratification, with the equilibrium surfaces (5.2), interacts with any part of the geostrophic flow that does not follow circular paths centred on the $z$-axis. This interaction, for $S \ll 1$, can be demonstrated in a manner similar to that used previously to derive (4.3). Taking the vertical average of the dot product of $\hat{\mathbf{k}}$ with the curl of (5.3$a$) and applying boundary condition (2.4), we obtain

$$\frac{\partial}{\partial t}\hat{\mathbf{k}}\,.\,\langle\nabla\times\mathbf{q}\rangle = S^{\frac{1}{2}}F\frac{\partial}{\partial\theta}\langle T\rangle. \tag{5.4}$$

If $S = 0$ or if $F = 0$, (5.4) expresses, as expected, the time independence of the vertically averaged $z$-component of vorticity. By integrating (5.4), with respect to $\theta$, around a closed path at a fixed value of $r$, we find

$$\frac{\partial}{\partial t}\int_0^{2\pi} \hat{\mathbf{k}}.\langle\nabla\times\mathbf{q}\rangle\, d\theta = 0. \tag{5.5}$$

We can reason, therefore, that the temporally conserved axisymmetric component of $\hat{\mathbf{k}}.\langle\nabla\times\mathbf{q}\rangle$ is 'carried' by a steady geostrophic flow. The non-axisymmetric component will, however, be time-dependent, and, to find an equation for this part, we continue by substituting the vertical average of (5.3$b$) into the time derivative of (5.4), to obtain, after substituting the expansion (3.3) (with modification (4.4)), the relation

$$\frac{\partial^2}{\partial\tau^2}\hat{\mathbf{k}}.\nabla\times\mathbf{q}_{g0} = F^2 r\frac{\partial}{\partial\theta}(\mathbf{q}_{g0}.\hat{\mathbf{r}}). \tag{5.6}$$

The velocity $\mathbf{q}_{g0}$ satisfies (3.9), and is given, in terms of $p_{g0}$, by (3.4). In addition, we will assume that the steady part of the geostrophic flow, satisfying (5.5), is written as a separate component.

Equation (5.6) expresses the basic balance for unsteady motions on a $\tau = S^{\frac{1}{2}}t$ time scale and describes the motion of weak stratification waves in this situation. Substituting (3.4) into (5.6), we obtain the equation for the pressure,

$$\frac{\partial^2}{\partial\tau^2}\nabla^2_H p_{g0} = -F^2\frac{\partial^2 p_{g0}}{\partial\theta^2}, \tag{5.7$a$}$$

where, as in §4.2, the boundary condition (2.4) on the velocity results in the condition,

$$p_{g0}(r = r_0) = 0. \tag{5.7$b$}$$

The eigenfunction solutions to (5.7$a, b$) are

$$p_{g0km} = J_k(\xi_{km}r/r_0)\exp[i(k\theta+\sigma_{km}\tau)], \tag{5.8$a$}$$

$$\sigma_{km} = \pm|k|\, r_0 F/\xi_{km}, \tag{5.8$b$}$$

where $k = \pm 1, \pm 2, \ldots$, and where $\xi_{mk}$ is the $m$th positive root of $J_k(\xi_{km}) = 0$.

A point of interest, in this particular problem, is that the restoring force for the waves varies with $r$ and decreases as the centre of the cylinder is approached. Noting the similarity of (5.7$a$) to the equation governing the propagation of ($z$-independent) internal waves, with stratification in the $r$ direction and with a *variable* Brunt–Väisälä frequency $rFN$, and recalling the properties of plane internal waves, we should expect Brunt–Väisälä trapping, for modes with wavelengths small compared with the cylinder dimensions, in an annulus of fluid bounded by the outer side wall of the cylinder and by an inner surface at some radius determined by the mode frequency. If the equivalent Brunt–Väisälä frequency is $rFN$, then we would expect the inner radius to be given by

$$r_C = |\sigma_D|/FN,$$

where $\sigma_D$ is the dimensional frequency.

This behaviour is, in fact, included in the solution (5.8). An examination of the asymptotic behaviour of the Bessel functions for large values of $k$ (Jeffreys & Jeffreys 1966, § 21.06) shows that the functions $J_k$ in (5.8) have an oscillatory behaviour for $r > r_C$, where $r_C = |k| r_0/\xi_{km} = |\sigma_{km}|/F$ ($r_C = |\sigma_{Dkm}|/FN$ in terms of the dimensional frequency), and an exponentially decaying behaviour for $r < r_C$. The critical radius $r_C = |\sigma_{km}|/F$ clearly represents a ray caustic for the waves of frequency $\sigma_{km}$ which are, therefore, trapped in the region $r_C \leqslant r \leqslant r_0$.

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## REFERENCES

Barcilon, V. 1968 Axisymmetric inertial oscillations of a rotating ring of fluid. *Mathematika*, **15**, 93–102.

Cole, J. D. 1968 *Perturbation Methods in Applied Mathematics*. Blaisdell.

Greenspan, H. P. 1965 On the general theory of contained rotating fluid motions. *J. Fluid Mech.* **22**, 449–462.

Greenspan, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

Greenspan, H. P. 1969 On the inviscid theory of rotating fluids. *Studies in Appl. Math.* **48**, 19–28.

Høiland, E. 1962 Discussion of a hyperbolic equation relating to inertia and gravitational fluid oscillations. *Geofys. Publ.* **24**, 211–227.

Howard, L. N. 1968 Notes on the 1968 Summer Program in Geophysical Fluid Dynamics, The Woods Hole Oceanographic Institution, Ref. no. 68–72, vol. I, 41–113.

Howard, L. N. & Siegmann, W. L. 1969 On the initial value problem for rotating stratified flow. *Studies in Appl. Math.* **48**, 153–169.

Jeffreys, H. & Jeffreys, B. S. 1966 *Methods of Mathematical Physics* (3rd edn.). Cambridge University Press.

Pedlosky, J. & Greenspan, H. P. 1967 A simple laboratory model for the oceanic circulation. *J. Fluid Mech.* **27**, 291–304.

Siegmann, W. L. 1968 Time-dependent motions in rotating stratified fluids. Ph.D. Thesis, M.I.T.